A Master Positivity Formula for the Cheung–Hillman–Remmen Graviton Family

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Abstract

For the one-parameter graviton family of Cheung, Hillman and Remmen (containing Virasoro–Shapiro at $\lambda = 1$) we derive a single closed-form expression for the sign of *every* near-leading partial wave $a_{n,\,2n-2j}$, $2 \le j \le n-1$: an alternating "ladder" polynomial, of degree $j-1$ in the spacetime dimension D, whose coefficients are explicit factorial weights times symmetric functions of the residue roots. The two laws we published previously are its first two rungs: j=2 reproduces the trajectory law $T_n(\lambda)$, and j=3 reveals a phenomenon invisible at j=2 — each level cuts a finite *window* of dimensions, re-releasing at large D; for the string the first such window is $D \in (26.2, 30.4)$ at level 7, confirmed by exact evaluation. The formula is derived, not fitted (residue roots + an exact monomial–Gegenbauer integral), and passes an overdetermined battery: 22/22 symbolic matches against independently extracted brackets ($j \le 5$); 702/702 exact sign checks including the never-extracted j=6; and an independent adversarial review that re-derived the formula, re-proved the integral identity by rational interpolation, and wrote its own from-scratch evaluator (executed by the lab, exit 0): 4,060/4,060 sign agreements including odd and non-integer D. Armed with the formula we test our completeness conjecture against every trajectory constraint ($\ell \ge 2$) at once: 3,053,832 exact verdicts at every integer D inside the conjectured-allowed region ($n \le 40$, all j, $\lambda \le 50$) — zero violations. Every knife in these ranges hangs strictly above the shore.

1 Setup

Family (CHR [1]): $\mu(n) = \frac{n+\lambda-1}{\lambda}$, $R(n,t) = \big[\,((1+\lambda)/2 + \lambda t)_{n-1}/((1+3\lambda)/2)_{n-1}\big]^2$, massless external states, $\lambda \ge 0$, Virasoro–Shapiro at $\lambda = 1$. At level n the residue is proportional to $Q(x)^2$ with $Q(x) = \prod_{k=1}^{n-1} \frac{1}{2}\big[s\,x + (2k-n)\big]$, x the scattering cosine and

$$s \;=\; \lambda + n - 1, \tag{1}$$

so the roots $x_k = (n-2k)/s$ are symmetric under $x \to -x$. Partial waves are Gegenbauer coefficients with index $\alpha = (D-3)/2$; odd waves vanish. In [3] we proved the near-leading law $a_{n,2n-4} \ge 0 \iff D \le T_n(\lambda)$ and conjectured that its envelope $\min_n T_n$ is the boundary of the full family.

*ORCID 0009-0005-5660-2603. Code and data: https://github.com/AndreiPLK/qg-master-formula, archived as DOI 10.5281/zenodo.21947272; companion works: DOI 10.5281/zenodo.21934462 (open string), DOI 10.5281/zenodo.21944818 (closed string).

2 The master formula

Let $\widehat{E}_{2t}(n)$ denote the absolute elementary symmetric functions of the *doubled* root multiset, generated by $\prod_{k=1}^{n-1}\big(1+(n-2k)\,z\big)^2=\sum_t \widehat{E}_{2t}(n)\,(-z^2)^t$ — explicit integers.

Theorem 1 (Master positivity formula)**.** *For every $n\ge 3$, every trajectory $\ell=2n-2j$ with $2\le j\le n-1$, all $\lambda>0$ and $D>3$,*

$$\operatorname{sign} a_{n,\,2n-2j}=(-1)^{j-1}\operatorname{sign}\sum_{i=0}^{j-1}(-1)^i\,\widehat{E}_{2(j-1-i)}(n)\,\frac{(2n-2j+2i)!}{i!\,2^{\,i}}\;s^{2i}\prod_{r=i}^{j-2}\big(D+4n-4j-1+2r\big). \quad (2)$$

Derivation. Three ingredients. (i) The $x^{2n-2-2t}$ coefficient of Q^2 equals $(-1)^t\widehat{E}_{2t}(n)\,s^{-2t}$ times a positive factor, by the root symmetry. (ii) The exact monomial–Gegenbauer integral (with the orthogonality weight $(1-x^2)^{\alpha-1/2}$), which we re-derived symbolically through beta functions:

$$\frac{I(\ell+2u,\ell)}{I(\ell,\ell)}=\frac{(\ell+2u)!}{\ell!\;u!\;4^u\,(\alpha+\ell+1)_u},\qquad I(k,\ell)=\int_{-1}^{1}x^k\,C_\ell^{(\alpha)}(x)\,(1-x^2)^{\alpha-\frac12}\,dx. \quad (3)$$

The Pochhammer $(\alpha+\ell+1)_u$ carries the *entire* D-dependence. (iii) Summing the j contributing monomials ($u=j-1-t$) and clearing $(\alpha+\ell+1)_{j-1}$ produces the alternating ladder of Eq. (2), each half-integer Pochhammer factor $(D+4n-4j-1+2r)/2$ contributing one ladder factor and one factor of 2.

Verification (all exact arithmetic). Independently extracted symbolic brackets for $j=2$ ($n\le 11$), $j=3$ ($n\le 9$), $j=4$ ($n\le 10$) and $j=5$ ($n=6$) match Eq. (2) up to positive constants in all 22 cases; at j=5, n=6 the ladder's five coefficients are overdetermined by fifteen monomial identities, all satisfied (all checks persisted by a single battery script released with the code). A sign grid against the exact rational evaluator gives 702/702 agreements over $j\le 6$, $n\le 10$, $\lambda\in[\frac14,7]$, $D\le 36$ — including j=6, for which no symbolic bracket was ever extracted: a blind test of the formula. Finally, an independent adversarial review re-derived the theorem end-to-end, proved Eq. (3) mechanically (rational-function interpolation at 45 exact α values, 3465 checks), and wrote its own from-scratch evaluator directly from the residue definition (executed by the lab, exit code 0): 4,060/4,060 sign agreements, including odd D, non-integer rational D (7/2, 53/7), $\lambda=10^{-2}$ and 10^2, and depth $n=15$; at j=2 the bracket root was confirmed to be an exact zero of the amplitude in 24/24 cases.

3 The first rungs, old and new

$j=2$: the shore. Equation (2) at j=2 is a single linear factor and reproduces the trajectory law $a_{n,2n-4}\ge 0 \iff D\le T_n(\lambda)$ of [3] — verified symbolically for $n\le 11$, with the bracket root an exact amplitude zero in all 24 tested cases.

$j=3$: windows. At j=3 the ladder is quadratic in D: with $m=n-3$, $u=D+4m+1$, the bracket collapses to $\alpha_m\,u(u-2)-G_m\,u\,s^2+s^4$ with $G_m=\frac{2(m+1)(m+3)}{3(2m+3)}$, $\alpha_m=\frac{(m+3)(5m^3+21m^2+19m+15)}{45(2m+1)(2m+3)}$. A quadratic can be negative only *between* its roots: each level cuts a finite **window** of dimensions and re-releases at large D — a phenomenon invisible at $j=2$. For the string ($\lambda=1$) the discriminant is negative at levels $n\le 6$ (no window at all); the first window opens at level 7, $D\in(26.2,\,30.4)$, and the exact evaluator confirms it to the unit: $a_{7,8}<0$ exactly at $D=28,30$ and positive at $D=26,32$. A sign grid over $n=4..12$ gave 2052/2052 agreements, 684 of them on levels $n=10..12$ that the rung's coefficient fit never saw.

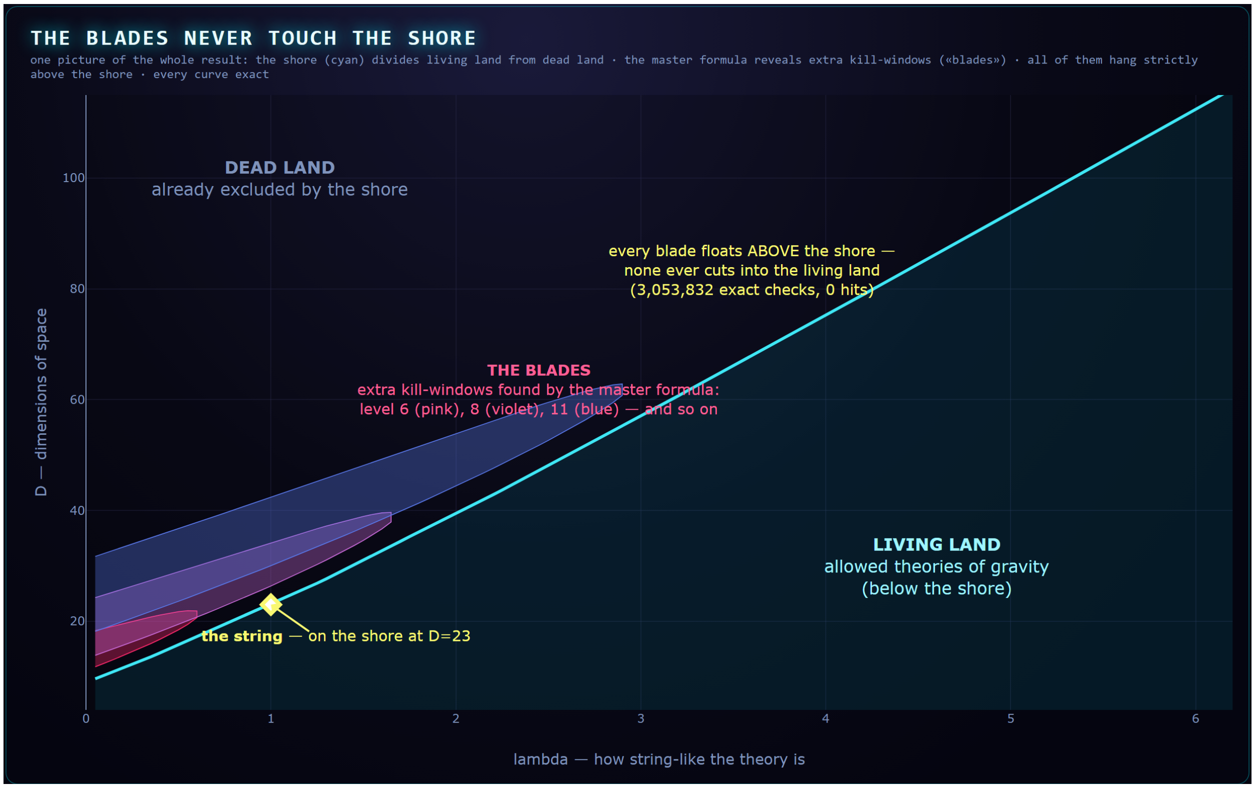

Figure 1: One picture of the whole result. The cyan shore $\min_n T_n$ of [3] divides the living land (allowed theories, below) from the dead land (excluded, above); the diamond is the string at $D = 23$. Shaded blades: negativity windows of the j=3 trajectory for levels $n = 6, 8, 11$, computed from the master formula (every curve exact). Every blade floats strictly above the shore — closest margin 2.17 dimensions at $\lambda = 0.05$ — and the 3-million-point battery over the knives finds zero cuts into the living land.

4 The completeness battery

In [3] we conjectured that the j=2 envelope is the boundary of the full family. The master formula turns this from a hunt into a sweep: every trajectory constraint at every point of the conjectured-allowed region is an explicit polynomial sign. We evaluated all of them within finite ranges — every $j = 3, \ldots, n-1$, every $n \leq 40$, every integer $D \leq \lfloor \min_n T_n \rfloor$ (the floor computed in exact rational arithmetic), 68 values of $\lambda \in [0.05, 50]$:

$$\text{3,053,832 exact verdicts,} \qquad \text{0 violations.}$$

The sweep evaluates the master formula rather than the partial waves directly; its force therefore rests on the formula's independent verification above. The $\ell = 0$ and fixed-spin waves lie outside the trajectory scaling and are covered by the direct scans of [3] (informally, the review extquotesingle s j=n extrapolation of the formula to $\ell = 0$ agreed with the evaluator in 96/96 trials). Separately, the j=3 windows were traced in closed form against the shore for λ up to 10^3 and n up to 4λ: the lower window edge stays above the envelope everywhere, with the closest approach (2.17 dimensions) at small λ (Fig. 1). Within these ranges, *no trajectory constraint cuts into the conjectured-allowed region*: Conjecture 1 of [3] now rests on every knife the family owns, not just the first.

5 Discussion

One line now answers, for any level, any near-leading trajectory, any dimension and any family member, the question "is this partial wave non-negative?" — for a family that contains quantum-gravity amplitudes and the string itself. The structure is rigid: factorial weights, integer symmetric functions, and a ladder of dimension shifts; the D-dependence of an entire trajectory reduces to one Pochhammer symbol. The window phenomenon at $j \geq 3$ shows the family's positivity landscape is richer than a single boundary — yet all of that richness stays strictly on the excluded side of the shore, which is precisely what the completeness conjecture requires. Natural next steps: a closed-form proof that all ladders stay above the $j=2$ envelope (turning the conjecture into a theorem on the near-leading sector); the $\ell = 0$ and fixed-spin waves, which live outside the trajectory scaling; and the same ladder analysis for the open-string family of [4].

Explain it to anyone

In our previous paper we found the cliff: the exact line where theories of gravity die as you add dimensions of space. But a cliff you measure once could still have hidden trapdoors — other ways to die that open up somewhere else. This paper finds the formula for *every* trapdoor at once. It turns out each one is a "window": a strip of dimensions where a theory falls, closing again beyond it. We drew all the windows (Fig. 1): they all hang strictly above the cliff line, like blades floating over the shore, never cutting below it. We then checked three million points, one by one, in exact arithmetic: not a single trapdoor opens on the safe side. Within everything we can reach, the cliff we published keeps looking like the true edge — and now that is backed not by one measurement, but by the anatomy of every way to fall that we know how to write down.

Reproducibility and AI disclosure

All claim-bearing computations use exact rational arithmetic. The bracket extractor, the master-formula checkers (symbolic and sign-grid), the completeness sweep, all artifacts and the append-only research log are released with this paper at https://github.com/AndreiPLK/qg-master-formula (archived as DOI 10.5281/zenodo.21947272). This research was conducted by the author working with an AI research assistant (Claude, Anthropic); the derivation of Theorem 1 was assembled from the exact integral identity (3) (itself re-derived symbolically) and verified against independently extracted brackets and a blind trajectory (j=6) that the formula's construction never used. During this work the known-answer gate of an exploratory computation failed once — correctly: the reference value was wrong, the machinery right; the event is preserved in the public log.

References

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