The Island Has Edges: Exact Boundary Laws for Unitary Deformations of the Veneziano Amplitude

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Abstract

The three-parameter family (q, r, w) of Veneziano deformations constructed by Cheung, Hillman and Remmen from level truncation and crossing constraints is restricted by partial-wave positivity to an "island" in parameter space, previously mapped numerically at level depth $n \leq 10$. We derive exact closed-form sign laws for the near-leading Regge partial waves of this family at $q = 1$ and obtain: (i) nothing survives left of the line $r = -(1 + \mu_0)/2$ — an exclusion valid at every level and in every spacetime dimension $D > 3$ (for $\mu_0 \leq 0$ the allowed region extends up to this line in our scans; for $\mu_0 > 0$ threshold constraints cut strictly inside it); (ii) the erosion of finite-depth islands is explained cell by cell by a single linear-in-n law, which also removes 176 false-positive points from a fine-grained depth-20 scan; (iii) the remaining boundary is cut by an explicit finite set of low-level algebraic curves whose D-dependence is a $1/(D - 1)$ tightening, explaining analytically why the island shrinks with dimension while its exclusion edge stays pinned; and (iv) the resulting analytic description — conjecturally complete — reproduces 11,994 exact-rational-arithmetic verdicts across seven mass shifts with zero unexplained mismatches. Beyond the exclusion edge, every odd near-leading trajectory with $k \leq 7$ eventually turns negative at a threshold our closed forms predict exactly, while even trajectories remain positive. For $q > 1$ we measure the exact depth at which unitarity excludes the deformation, $n_{\text{crit}} \simeq 1.1\,(q - 1)^{-1/2}$, and derive the $-1/2$ exponent from the quadratic growth of the relative q-correction. On the $w = 0$ slice our edge law reproduces the asymptotic contour result of Mansfield and Spradlin; to our knowledge — based on a survey of the works citing CHR indexed in INSPIRE — the $w \neq 0$ region has not been treated analytically before.

1 Introduction

The modern S-matrix bootstrap asks which scattering amplitudes are consistent with unitarity, crossing, and assumed high-energy behavior — and, in particular, how special string amplitudes are within that space. Cheung, Hillman and Remmen (CHR) [1] showed that elevating the Veneziano amplitude's *level truncation* property to an axiom, together with crossing at truncation points, forces a three-parameter family of dual resonant amplitudes labeled (q, r, w), with spectrum $\mu(n) = [n]_q$; the Veneziano amplitude sits at $(1, 0, 0)$. Imposing partial-wave positivity, CHR mapped numerically (levels $n \leq 10$, $D = 4$) a broad allowed region — an island — in the (r, w) plane.

Two natural questions were left open. First, what happens to the island as the positivity depth n grows — does it erode away, or stabilize, and what is its true boundary? Second, positivity was

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imposed numerically; is the boundary itself computable in closed form? This paper answers both for $q = 1$ (with a quantitative excursion into $q > 1$), using exact rational arithmetic throughout and a derivation strategy simple enough to state in one line: *track the top coefficients of the level-n residue polynomial through the partial-wave projection.*

Our results turn the island from a numerical observation into an analytically described object, with every claim below either derived in closed form, verified against exact scans, or explicitly labeled as a conjecture.

2 Setup

At $q = 1$ the level-n residue of the CHR amplitude (their Eq. (16)) reduces to

$$R(n,t) \;=\; \frac{(2+r+t)_{n-1}\,\big(1+r+t+w\big)\big(1+n+r+w\big)}{(2+r)_n\,(1+r+w)}, \tag{1}$$

a degree-n polynomial in t with roots $t = -(2 + r + k)$, $k = 0, \dots, n-2$, and $t = -(1 + r + w)$. With external masses μ_0 (spectrum $\mu(n) = n + \mu_0$) the scattering-angle substitution at level n is $t = \alpha(x - 1) - \mu_0$ with $\alpha = (n - 3\mu_0)/2$, $x = \cos\theta$. In D spacetime dimensions the partial-wave coefficients $a_{n,\ell}$ are the Gegenbauer coefficients ($\alpha_G = \frac{D-3}{2}$; Legendre for $D = 4$) of R restricted to level n. Unitarity requires $a_{n,\ell} \geq 0$ for all above-threshold levels ($\mu(n) \geq 4\mu_0$, per CHR).

All numerics below use exact rational arithmetic (Python `Fraction`); the evaluator was validated by two independent routes (the CHR closed form (A6) versus a first-principles residue-plus-projection computation), which agree at every tested (n, ℓ, r, w, μ_0) up to a positive ℓ-dependent normalization. The *domain* for every sign statement below is $2 + r > 0$ and $1 + r + w > 0$; outside it the positive prefactor in Eq. (1) flips sign, and the corresponding full sign laws were checked separately.

3 The edge theorem

Theorem 1 (Left edge)**.** *For $q = 1$, $D > 3$, $n > 3\mu_0$, and (r, w) in the domain,*

$$\operatorname{sgn} a_{n,n-1} \;=\; \operatorname{sgn}\Big[\,n\Big(r + \tfrac{1+\mu_0}{2}\Big) + w\,\Big]. \tag{2}$$

Proof sketch. Write the monic part of Eq. (1) as $p(t) = t^n + A\,t^{n-1} + \dots$ with $A = (n-1)(2+r) + \binom{n-1}{2} + (1+r+w)$. Substituting $t = \alpha x + \beta$ ($\beta = -\alpha - \mu_0$), the coefficient of x^{n-1} is $\alpha^{n-1}(A + n\beta)$, and $A + n\beta = n\big(r + \frac{1+\mu_0}{2}\big) + w$. By parity and degree, x^n projects only onto $C_n^{(\alpha_G)}, C_{n-2}^{(\alpha_G)}, \dots$ and monomials below x^{n-1} of the correct parity have degree $< n - 1$; hence only the x^{n-1} monomial feeds $C_{n-1}^{(\alpha_G)}$, with a positive projection constant for every $D > 3$. The prefactor in Eq. (1) is positive in the domain, and $\alpha > 0$ above threshold. (At $n = 3\mu_0$ exactly, $\alpha = 0$ and $a_{n,n-1}$ vanishes identically.) □

Consequences. Nothing with $r < -(1 + \mu_0)/2$ and $w > 0$ survives: positivity fails at depth $n > w/|r + \frac{1+\mu_0}{2}|$. The edge is *dimension-universal*: the x^{n-1} coefficient carries no D-dependence. Finite-depth scans overstate the island in a sliver next to the edge, with erosion depth inversely proportional to the distance.

Verification. Eight "razor" predictions of exact zeros were confirmed in exact arithmetic, including at points never scanned before (e.g. $(r, w) = (-13/25, 1/2)$: predicted zero at $n = 25$; measured $a_{24,23} > 0$, $a_{25,24} = 0$, $a_{26,25} < 0$), at $\mu_0 = \pm 3/5$, and at $D = 6, 10$. Sixty random (n, r, w)

draws confirm the full sign law including prefactor flips outside the domain. An independent adversarial review re-derived every step and its six-attack falsification script (published with the code) found no counterexample.

The nine casualties. In depth-graded scans at $\mu_0 = 0$, exactly nine cells vanish between $n \le 10$ and $n \le 20$, all at $r = -3/5$: the law predicts death at $n = 10w+1$, matching the measured first-negative list cell by cell ($w = 1 \to (11, 10)$, …, $w = 9/5 \to (19, 18)$) and explaining why no casualties occur at any greater depth. The same phenomenon recurs across the μ_0 stack: thirty map-allowed cells at distance 0.1 from the shifted edges are doomed at predicted depths; four were verified directly, each with the marginal exact zero $a_{10w,10w-1} = 0$ at the preceding level.

4 Trajectory laws and the odd/even dichotomy

The same top-coefficient method yields, at $\mu_0 = 0$ in $D = 4$:

$$\operatorname{sgn} a_{n,n-2} = \operatorname{sgn}\Big[\, 12(2n-1)(1+r)(nr+2w) + n(n^2+5n-2) \,\Big], \tag{3}$$

$$\begin{aligned}\operatorname{sgn} a_{n,n-3} = \operatorname{sgn}\Big[\, & n^3(2r+1) + 16n^2r^3 + 24n^2r^2 + 14n^2r + 2n^2w + 3n^2 \\ & - 24nr^3 + 48nr^2w - 36nr^2 + 96nrw - 12nr + 54nw \\ & - 72r^2w - 144rw - 72w \,\Big].\end{aligned} \tag{4}$$

Eq. (3) was proven for all n (an exact polynomial identity; positive constant $24(2n-1)/(n-1)$); Eq. (4) was verified against brute force for $n = 4, \ldots, 9$ at random rational points. The cubic-in-n terms dominate at large n: for $r > -1/2$ both trajectories are eventually positive and can exclude only within a finite window of levels.

Empirically the pattern extends: for $k = 1, \ldots, 7$, beyond the edge every *odd* trajectory $a_{n,n-k}$ eventually turns negative — at finite thresholds that Eqs. (2)–(4) predict exactly (the $k = 3$ bracket predicts a sign change at $n = 57$ for $(r, w) = (-0.52, 1)$; measured between 55 and 59) — while every *even* trajectory remains positive on both sides. The edge is thus witnessed by an infinite family of independent constraints; a general-k proof is left open.

5 The island, analytically

A census of first-negative pairs (n, ℓ) over all 714 excluded cells of the depth-40 map at $\mu_0 = 0$ shows that *every* binding constraint is either (i) a level $n \le 5$ constraint, (ii) the $\ell = n - 1$ ladder of Theorem 1, or (iii) a domain pole. This motivates (see Figs. 1–2):

Conjecture 1 (Complete characterization, $\mu_0 = 0$, $D = 4$)**.** *(r, w) is allowed at all depths iff all $a_{n,\ell} \ge 0$ for $n \le 5$ (twenty explicit algebraic curves, e.g. $a_{2,0}:\ 3(1 + r)(r + w) + 1 \ge 0$ and $a_{3,0}:\ 4r^3 + 4r^2w + 6r^2 + 8rw + 8r + 6w + 3 \ge 0$) and the ladder condition $\neg(r < -\frac{1}{2} \wedge w > 0)$ holds, within the domain.*

The conjecture reproduces 1369/1369 coarse cells (depth 40; the boundary cells re-tested at depth 80 with no change) and 2411/2411 fine boundary points at resolution 0.02 — after the ladder removes 176 false positives of the finite-depth scan — and extends across the μ_0 stack: with the binding window taken at the first above-threshold levels ($n_{\min} = \min\{n : n \ge 3\mu_0\}$), the analytic verdict matches all six $\mu_0 \ne 0$ maps up to precisely the thirty doomed cells described above. Total: 11,994 exact verdicts with zero *unexplained* mismatches (the thirty doomed cells are

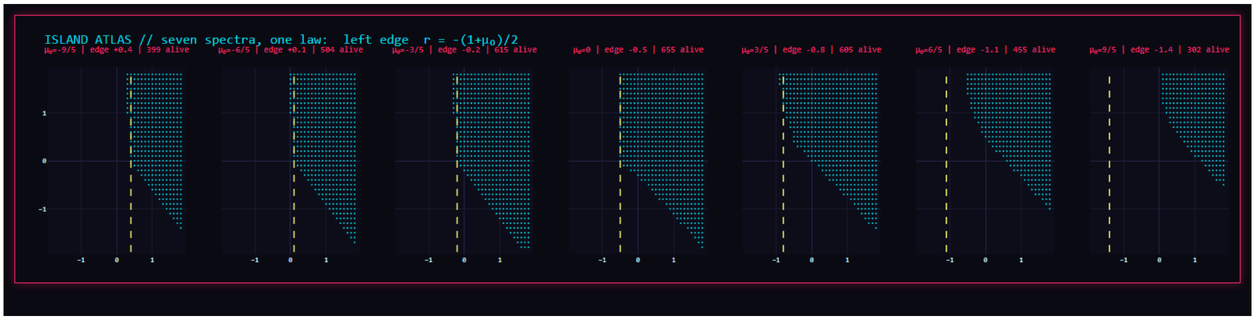

Figure 1: Positivity islands for seven mass shifts μ_0 (exact rational scans; cyan = allowed). Dashed line: the exclusion edge $r = -(1+\mu_0)/2$. For $\mu_0 \le 0$ the island reaches the line; for $\mu_0 > 0$ threshold constraints cut strictly inside it.

finite-depth artifacts of the $n \le 10$ maps, predicted by the ladder and verified by direct evaluation in four cases). What remains unproven is that no constraint with $\ell \le n-2$, $n > n_{\min}+4$ ever binds; supporting evidence includes the finite kill-windows of Eqs. (3)–(4) and fixed-spin positivity below.

Threshold structure at $\mu_0 > 0$. The dominant killers sit at the first above-threshold level ($n_{\min} = 2, 4, 6$ for $\mu_0 = 3/5, 6/5, 9/5$), predominantly the scalar channel: as $\alpha \to 0$ the residue degenerates toward pure $\ell = 0$. The scalar curve is explicit, e.g. $a_{2,0}(r, w, \mu_0) \propto 6\mu_0^2 + 6\mu_0 r + 3\mu_0 w - 3\mu_0 + 6r^2 + 6rw + 6r + 6w + 2$, factoring at $\mu_0 = 2/3$ as $(3r+4)(3r+3w+1)/9$.

Dimension dependence. With the positive factor $\sqrt{\pi}\,\Gamma(\frac{D}{2}-1)/\Gamma(\frac{D}{2}-\frac{1}{2})$ removed, $a_{1,0} \propto r + w + \frac{1}{2}$ is D-independent, while

$$a_{2,0} \;\propto\; (1+r)(r+w) + \frac{1}{D-1}, \tag{5}$$

and $a_{3,0}$ carries analogous $1/(D-1)$ coefficients. The island therefore shrinks with D through a $1/(D-1)$ tightening of its scalar curves — an analytic account of CHR's numerical observation — while its left edge stays pinned at $-(1+\mu_0)/2$ in every dimension.

Fixed-spin tails. Localizing the projection at $x = 1$ gives $a_{n,\ell} \sim (2\ell+1)\,C(r,w)/(n \ln n)$ with $C > 0$ across the domain; the $(2\ell+1)$ scaling agrees to three digits between $\ell = 0$ and $\ell = 2$ in exact numerics, with positivity checked up to $n = 200$ at representative points, while the constant converges logarithmically. This part is heuristic-plus-numeric.

6 The q-clock

For $q > 1$ CHR excluded the deformation asymptotically. Using their Eq. (16) directly (roots $\xi(k) = [-1-r-k]_q$ are bounded by $-1/(q-1)$ while the angle substitution grows as $[n]_q \sim q^n$), we scanned the q-Veneziano line ($r = w = 0$) in exact arithmetic and measured the depth at which the first negative partial wave appears:

$q-1$	0.001	0.005	0.01	0.02	0.05	0.1
n_{crit}	34	15	11	8	5	4

consistent with $n_{\text{crit}} \simeq 1.1\,(q-1)^{-1/2}$ over two decades (the $q = 1$ control is clean to $n = 25$; the killer is low spin, $\ell = 0, 1$, not the ladder). The exponent follows from the logarithmic derivative

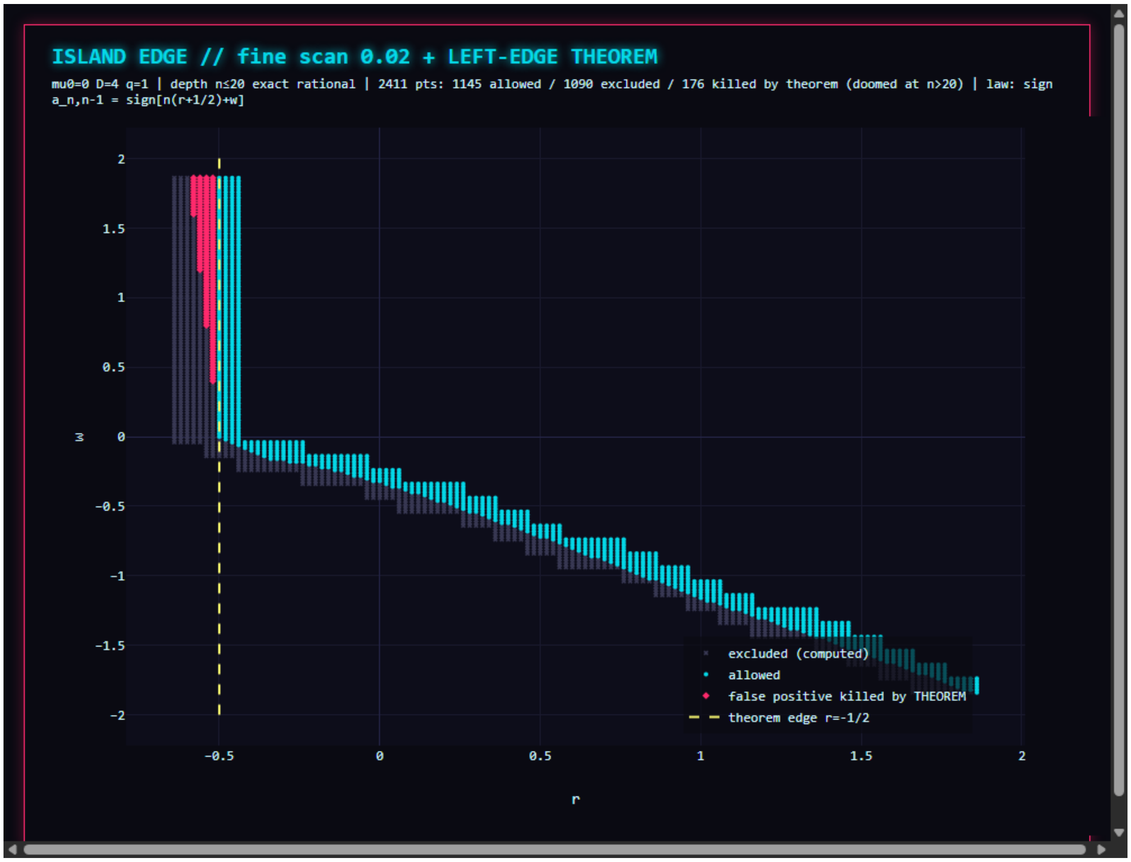

Figure 2: Fine boundary scan at resolution 0.02 ($\mu_0 = 0$, depth $n \le 20$). Cyan: allowed; grey: excluded by direct computation; pink diamonds: the 176 points admitted by the finite-depth scan but excluded by the ladder law, each with its predicted death depth; dashed line: $r = -1/2$.

(computed as an exact finite difference at step 10^{-6}) $-\partial_q \log a_{n,0}\big|_{q=1} \approx 0.3\, n^2$: the relative q-correction grows quadratically (the spectrum drift $[n]_q \approx n\big(1 + \frac{(n-1)(q-1)}{2}\big)$ accumulates), so the crossing occurs at $(q-1)\, n^2 = O(1)$. This yields a practical corollary: any depth-N scan necessarily admits $q-1 \lesssim (1.1/N)^2$, quantifying the q-tolerance of finite numerical bootstraps (e.g. $q-1 < 0.012$ at CHR's depth 10).

7 Relation to prior work

Mansfield and Spradlin [2] analyzed the $w = 0$ (hypergeometric) slice of this family with double-contour methods. Their odd-Δ Regge asymptotics carries the overall factor $(2r + m^2 + 1)$ — asymptotic positivity requiring exactly our edge line at $w = 0$. The two approaches agree where they overlap, while Theorem 1 is exact at every finite level and covers $w \neq 0$, which their analysis does not treat. Within the 49 works citing CHR indexed in INSPIRE (surveyed 2026-08-13/14) we found no other treatment of the $w \neq 0$ region or of the island's infinite-depth fate.

8 Discussion

Within CHR's explicitly stated assumptions, the boundary of the space of consistent Veneziano neighbors is no longer a numerical scan but a short list of explicit algebraic conditions, with one conjectural completeness statement that survived every falsification attempt we could construct (11,994 exact verdicts; adversarial review; thirty successful doomed-cell predictions; eight razor zeros). The natural next steps are a general-k proof of the odd-trajectory dichotomy, a rigorous fixed-spin asymptotic, and — toward quantum gravity — transplanting the top-coefficient method to families with double zeros in their residues (Virasoro–Shapiro-like), where CHR posed the corresponding uniqueness question for gravitational amplitudes.

Reproducibility and AI disclosure

All computations are exact-rational and scripted; the repository (github.com/AndreiPLK/qg-island-edges, DOI 10.5281/zenodo.21934462) contains the two-route evaluator, all maps and scans, the razor tests, the independent reviewer's attack script, and the append-only research log. This research was conducted by the author working with an AI research assistant (Claude, Anthropic); every scientific claim above was verified by deterministic code whose outputs are published, and the assistant's derivations passed an independent adversarial review recorded in the repository.

References

[1] C. Cheung, A. Hillman and G. N. Remmen, "Bootstrap Principle for the Spectrum and Scattering of Strings," Phys. Rev. Lett. 133, 251601 (2024), arXiv:2406.02665.

[2] G. Mansfield and M. Spradlin, "On Unitarity of the Hypergeometric Amplitude," JHEP 02 (2025) 145, arXiv:2409.09561.