The Shore of Closed-String Gravity: an Exact Unitarity Boundary for the Cheung–Hillman–Remmen Graviton Family

Andrei Pluzhnik*

August 2026

Abstract

The level-truncation bootstrap of Cheung, Hillman and Remmen produces a one-parameter family of graviton amplitudes containing the Virasoro–Shapiro amplitude at $\lambda = 1$, with residues that are perfect squares; partial-wave positivity was known to bound λ from below for $D \geq 9$, numerically. We derive the boundary in closed form. The near-leading even trajectory obeys $a_{n,2n-4} \geq 0 \iff D \leq T_n(\lambda)$ with $T_n(\lambda) = \frac{3(2n-3)}{n(n-2)}\big(\lambda^2 + (2n-2)\lambda + 1\big) + 2n$, verified exactly for $n \leq 14$ and against exact-rational scans. On this trajectory the boundary is the envelope $\min_n T_n$ — and we conjecture (with the falsification battery reported below) that it is the boundary of the full family. The shore is born at $(D, \lambda) = (9, 0)$ — explaining the onset observed by CHR — passes *exactly* through the string at $D = 23$ (so pure Virasoro–Shapiro first loses positivity on this trajectory at $D = 24$, via the level-4 spin-4 wave, matching exact scans), and straightens onto the exact asymptote $D = (12+4\sqrt{3})\,\lambda$ with active level $n^* \simeq \sqrt{3}\,\lambda$. Doomed-cell predictions of the envelope are confirmed by direct evaluation. We conjecture the envelope is the complete boundary and report the falsification battery it has survived.

1 Setup

Family (CHR, PRD 111, 086034): $\mu(n) = \frac{n+\lambda-1}{\lambda}$, $R(n,t) = \big[\,((1+\lambda)/2 + \lambda t)_{n-1}/((1+3\lambda)/2)_{n-1}\big]^2$, massless external states ($s+t+u = 0$), $\lambda \geq 0$, VS at $\lambda = 1$. Partial waves: Gegenbauer coefficients at $s = \mu(n)$, $t = -\mu(n)(1-x)/2$; odd waves vanish by the $t \leftrightarrow u$ symmetry of the squared residue. All computations exact-rational; evaluator validated against CHR's reported behavior (D=4 positivity for all λ; onset of the lower bound at $D \geq 9$).

2 The trajectory law

Theorem 1 (Near-leading even trajectory)**.** *For the residues above, $\lambda > 0$ and $D > 3$,*

$$a_{n,2n-4} \;\geq\; 0 \quad\iff\quad D \;\leq\; T_n(\lambda), \qquad T_n(\lambda) = \frac{3(2n-3)}{n(n-2)}\big(\lambda^2 + (2n-2)\lambda + 1\big) + 2n. \qquad (1)$$

Derivation. Write $\lambda\big(t - \xi(k)\big) = \frac{1}{2}\big[(n+\lambda-1)x + (2k-n)\big]$ at level n; the residue is the square of $Q(x) = \prod_{k=1}^{n-1} \lambda(t - \xi(k))$ up to a positive factor. The roots $x_k = (n-2k)/(n+\lambda-1)$ are symmetric under $x \to -x$, so Q has definite parity: odd partial waves vanish identically, and

*ORCID 0009-0005-5660-2603. Code and data: https://github.com/AndreiPLK/qg-gravity-shore, archived as DOI 10.5281/zenodo.21944818; companion work: DOI 10.5281/zenodo.21934462.

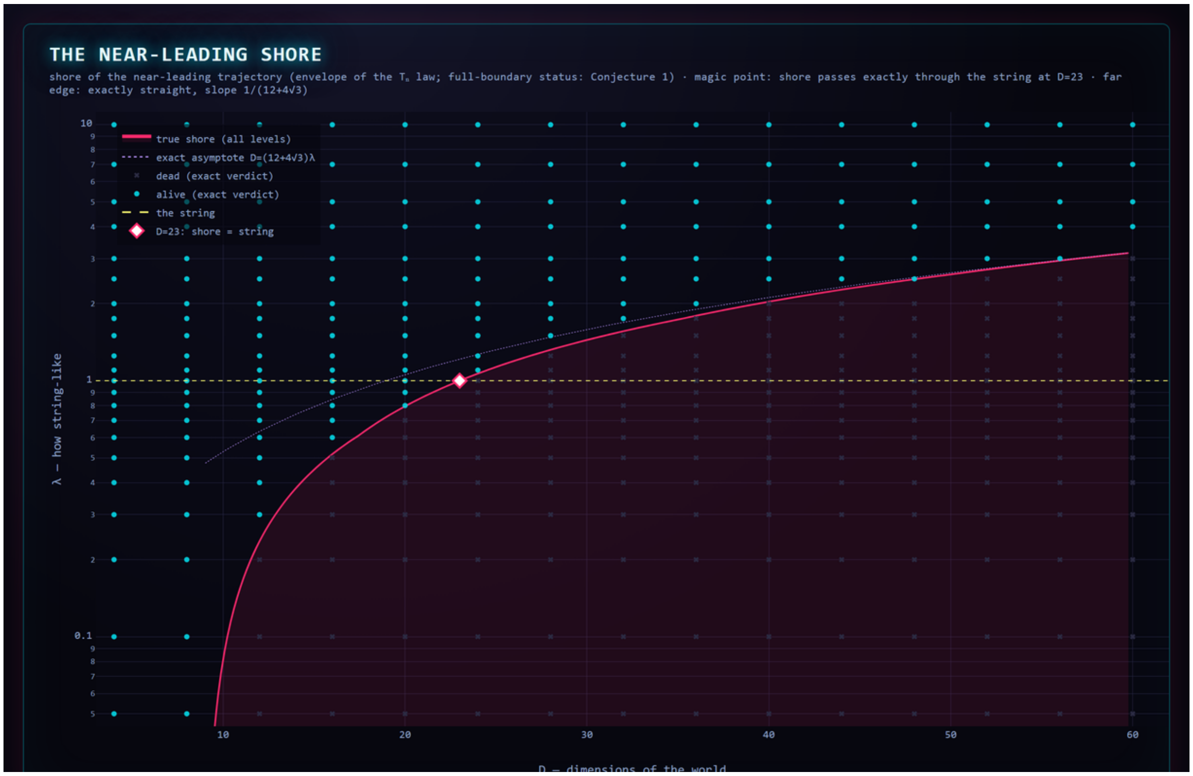

Figure 1: The boundary of the family: exact verdicts (cyan alive, grey dead; λ on a log scale), the analytic shore $\min_n T_n(\lambda)$, its straight asymptote, and the point $D = 23$ where the shore passes exactly through the Virasoro–Shapiro amplitude.

the subleading coefficient q_{n-2} of Q vanishes. Hence the x^{2n-4} coefficient of Q^2 is the *pure cross term* $2q_{n-1}q_{n-3} < 0$ — a naive term-by-term estimate would miss the edge entirely. Only x^{2n-2} and x^{2n-4} feed the Gegenbauer mode $\ell = 2n - 4$ (parity and degree), with the exact projection ratio $\rho(\ell, D) = \frac{(\ell+1)(\ell+2)}{2(D+2\ell-1)}$; assembling the two contributions and removing positive factors yields T_n. Using $e_2(\text{roots}) = -\frac{n(n-1)(n-2)}{6(\lambda+n-1)^2}$ one obtains the equivalent closed form $T_n = \frac{3(2n-3)(\lambda+n-1)^2}{n(n-2)} - (4n - 9)$.

Verification. The bracket was extracted independently by symbolic computation for every $n \leq 14$ and matches T_n identically (12/12); an independent adversarial review re-derived the theorem from scratch, and its falsification suite (published with the code) found no counterexample, including razor zeros at fractional dimensions $D^* = 271/4$ (n=5, λ=7/2) and $D^* = 57$ (n=6, λ=3) computed directly from the projection with symbolic D. Doomed-cell executions confirm predicted kill levels exactly (e.g. $(\lambda, D) = (3/2, 32)$ and $(2, 40)$: sign flip at $n = 4$ as predicted).

3 The envelope

The boundary implied by Theorem 1 is the envelope $\lambda_{\min}(D)$ defined by $D = \min_{n\geq 3} T_n(\lambda)$. Its structure:

Onset at $D = 9$. $T_n(0) = 2n + \frac{3(2n-3)}{n(n-2)}$ is minimized at $n = 3$ with $T_3(0) = 9$: the trajectory constrains nothing for $D \leq 9$ — reproducing, as a formula, the onset CHR observed numerically ($D \geq 9$).

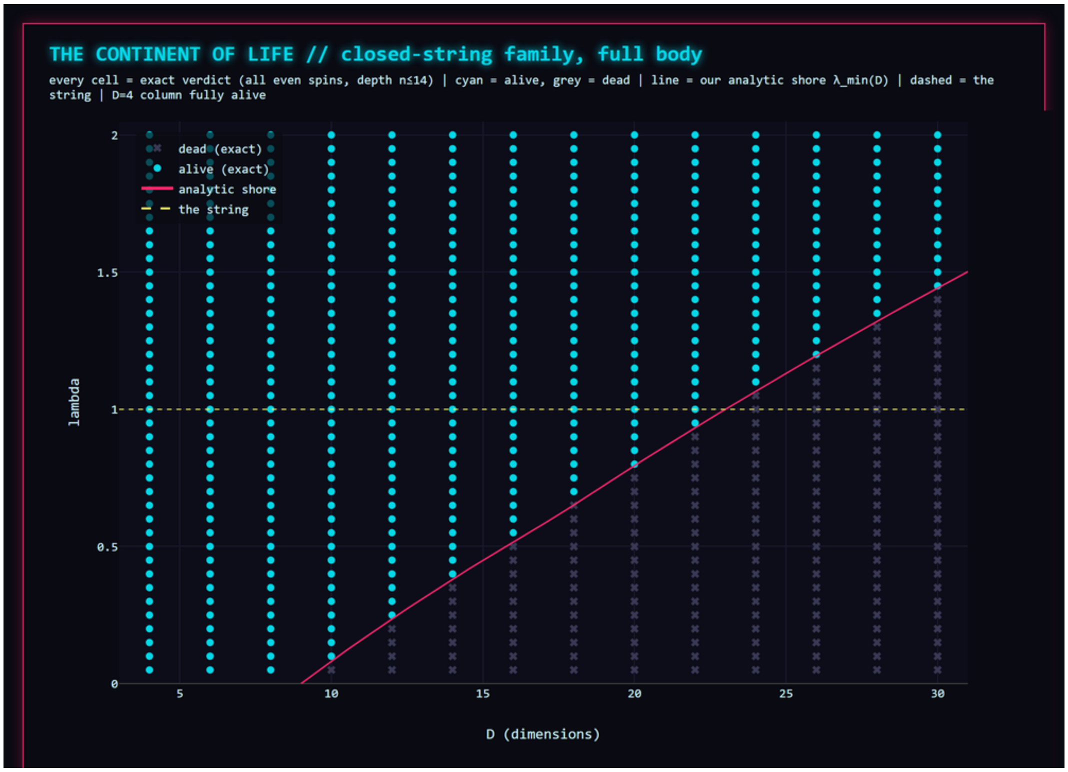

Figure 2: The allowed body at linear scale with the near-leading shore overlaid; every cell is an exact rational verdict (all even spins, depth 14).

The magic point. $T_n(1) = \frac{2(n^2+4n-9)}{n-2}$, whose integer minimum is 23, attained only at $n = 4$ (continuous minimum at $n = 2 + \sqrt{3}$). Hence $\lambda_{\min}(23) = 1$ *exactly*: the shore passes through the Virasoro–Shapiro point, and pure closed-string gravity first loses positivity on this trajectory at $D = 24$, through its level-4 spin-4 wave — matching the exact scan (first negative at $(n, \ell) = (4, 4)$ for $D = 24$, clean at $D = 22$ to depth 40).

The straight far edge. Minimizing T_n over continuous n at large λ gives the active level $n^* = \sqrt{3}\,\lambda$ and the exact asymptote $D = (12 + 4\sqrt{3})\lambda$; numerically $D/\lambda_{\min}$ rises $18.75 \to 18.92$ over $D = 60 \to 10^4$ against $12 + 4\sqrt{3} = 18.928\ldots$, with the residual $O(1/\lambda)$ oscillation caused by the discreteness of n^*.

4 Virasoro–Shapiro thresholds

For the string itself the theorem gives the threshold sequence $D_n(1) = 2(n^2 + 4n - 9)/(n - 2) =$ 24, 23, 24, $\frac{51}{2}$, $\frac{136}{5}$, ... — an exact refinement, on this trajectory, of the finite-level $D_{\text{crit}}(n)$ phenomenology known for Veneziano-type amplitudes. Because $D_n(1)$ grows linearly, the trajectory endangers VS only within a finite window of levels at any fixed D; the deep asymptotic bounds of the literature arise from other (low-spin) constraints and are complementary to the exact finite-level statement proven here.

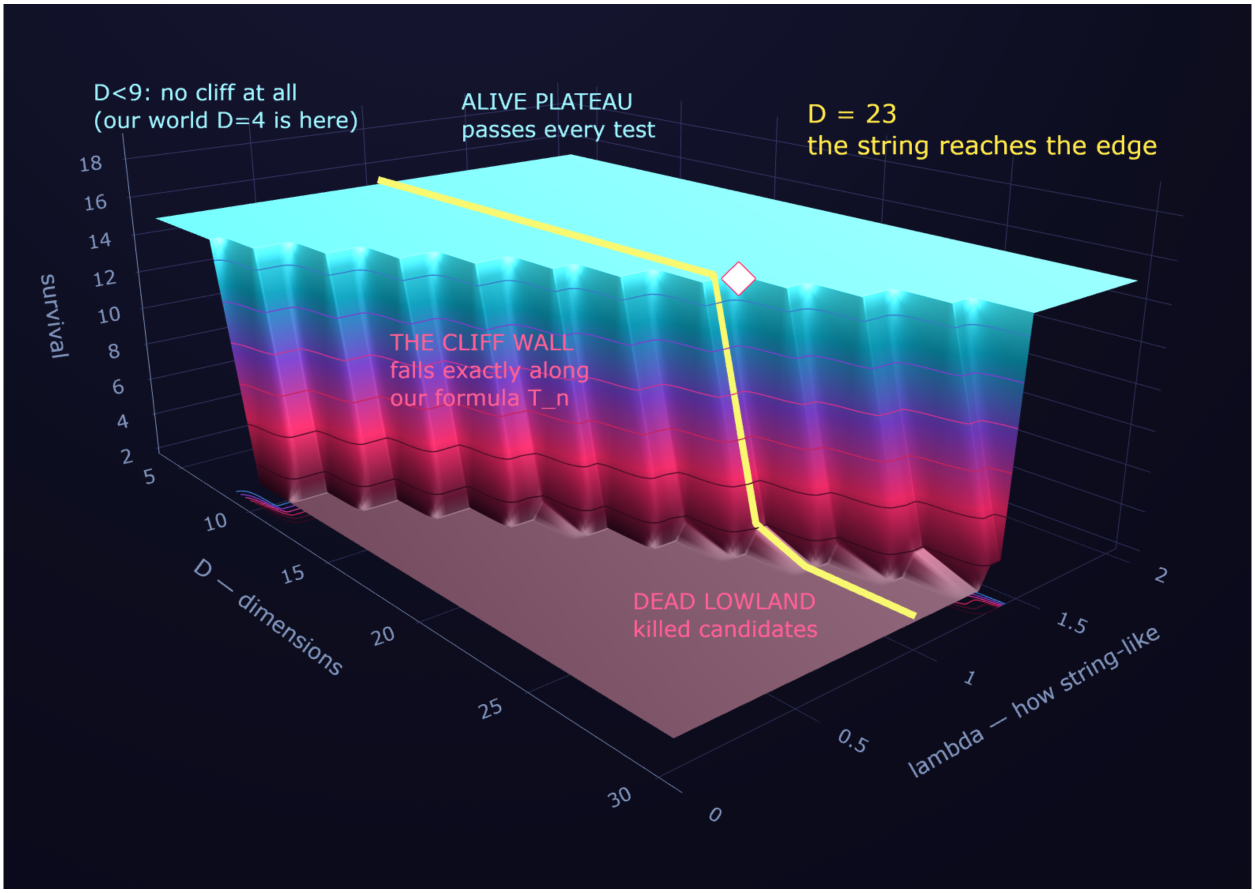

Figure 3: The same boundary as a landscape (every point an exact rational verdict). Height is the depth to which a candidate survives the positivity tests: the cyan plateau is the allowed region, the wall falls exactly along $\min_n T_n(\lambda)$, and the floor is the excluded region. The yellow lane is the string ($\lambda = 1$) walking the plateau as D grows; the white diamond marks $D = 23$, where the shore passes exactly through it. For $D < 9$ there is no wall at all.

5 Conjectured completeness and its battery

Conjecture 1. *A point (λ, D) of the family is consistent with partial-wave positivity at all levels iff $D \le \min_{n\ge3} T_n(\lambda)$.*

The conjecture is falsifiable: a single exact negative coefficient at a point with $D \le \min_n T_n$ refutes it. Its current battery: 494/494 agreements with exact full-spin scans (all even $\ell \le 2n-2$, depth 16, grid $\lambda \in [0.1, 10]$, $D \in [4, 40]$) with zero alarms of either type; two direct doomed-cell executions at the predicted level; and an independent adversarial review whose suite (seven attack batteries) found no counterexample. The reviewer-designated most plausible additional knife — the $\ell = 2n-6$ trajectory — has been hunted across stress zones hugging the conjectured boundary (34 values of λ, depth windows $[\min_n T_n - 3, \min_n T_n]$, $n \le 14$): zero alarms; further trajectories and low-spin stress tests are running as standing batteries. In $D = 4$ the model predicts (and scans confirm, 40/40 to depth 14 plus the trivial bound $T_n \ge T_n(0) \ge 9 > 4$) that the entire family survives: within these assumptions, four-dimensional gravity is *not* forced to be string-like at any finite depth we can reach.

6 Discussion

Together with the companion open-string results, a graded picture emerges: in $D = 4$ the entire λ-family survives every test we can run — at this level of assumptions, four-dimensional graviton scattering is not forced to be string-like — while as D grows the shore rises and passes exactly through the Virasoro–Shapiro point at $D = 23$: the string becomes extremal precisely in the gravitational, high-dimensional setting. The method (top coefficients of residue polynomials through the partial-wave projection) transfers verbatim between families and is cheap: every number here was computed on a desktop in exact arithmetic. Natural next steps: closing Conjecture 1; the planar analogues CHR construct; and mapping the surviving $D = 4$ alternatives to their effective-field-theory coefficients, where gravitational-wave phenomenology could in principle discriminate them from the string.

Explain it to anyone

Imagine every possible theory of quantum gravity in this family as a hiker on a high plateau (Fig. 3). Walking east means adding spacetime dimensions. Our result is the exact map of the cliff at the plateau's edge: below nine dimensions there is no cliff and everyone is safe — including our own four-dimensional world; from nine dimensions on, a cliff appears, and we found the formula for exactly where it runs. The most famous hiker — string theory — walks the plateau and reaches the very edge of the cliff at exactly 23 dimensions: one more step and it falls. That the boundary of consistency passes exactly through the string, and only in high dimensions, is the whole story of this paper in one picture. For fun, Fig. 4 draws the companion open-string landscape the same way: there the allowed region is not a plateau but a solid ship's bow sailing through a sea of dead candidates.

Reproducibility and AI disclosure

All claim-bearing computations use exact rational arithmetic; the evaluator, the falsification suites (the adversarial reviewer's seven-battery script included), all scan artifacts, and the append-only research log are released with this paper at `https://github.com/AndreiPLK/qg-gravity-shore` (archived as DOI 10.5281/zenodo.21944818). This research was conducted by the author working with an AI research assistant (Claude, Anthropic); every derivation passed an independent adversarial review, and every number in this paper is regenerated by a single published battery script. During this work an implementation bug (a Pochhammer increment step, coinciding with the correct evaluator only at $\lambda = 1$) was found, disclosed and fixed in the public log; all results affected by it were voided and recomputed, and the $\lambda = 1$ results it could not affect were re-verified independently.

References

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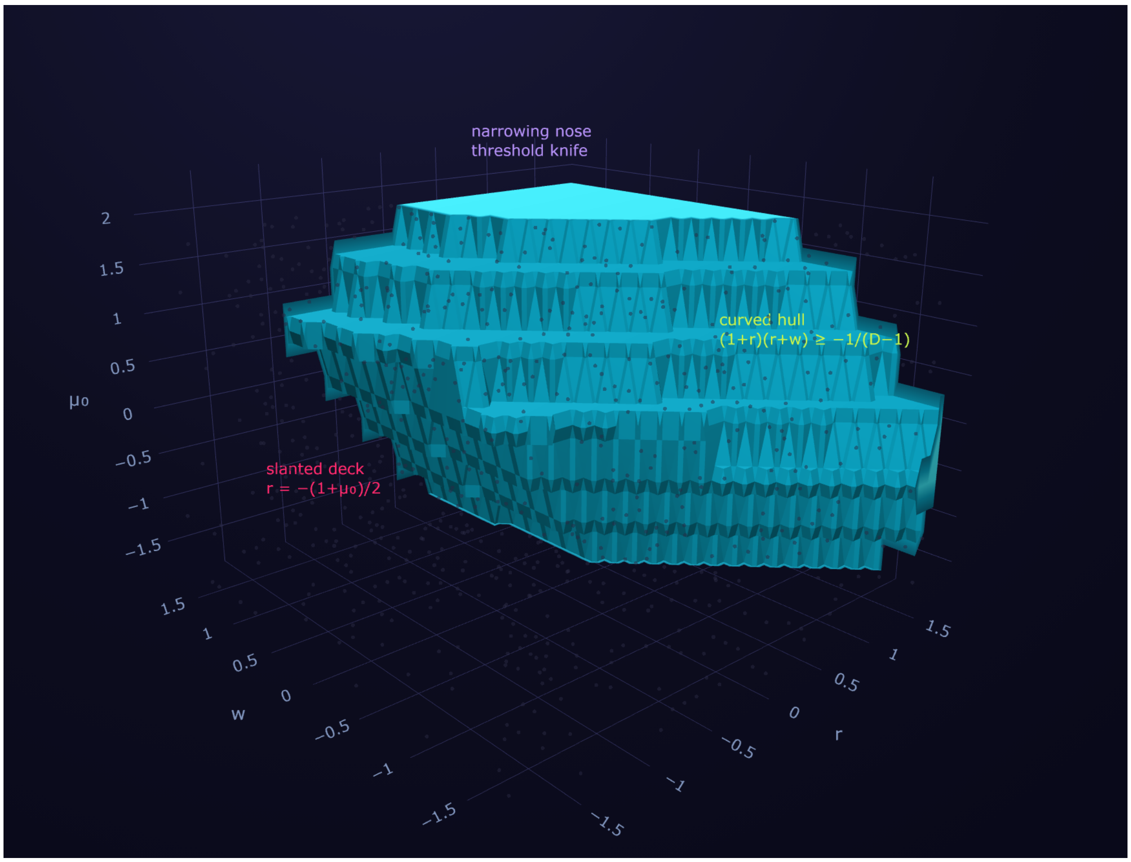

Figure 4: For fun: the companion *open-string* family [3] drawn in the same style — the solid of surviving candidates in (r, w, μ_0), every voxel an exact rational verdict. The slanted deck is the edge law, the curved hull the hyperbola $(1+r)(r+w) \geq -1/(D-1)$, and the narrowing nose the threshold knife of that paper.