The Blades Never Touch the Shore: a Proof for the Second Knife of Closed-String Gravity

Andrei Pluzhnik*

August 2026

Abstract

For the graviton family of Cheung, Hillman and Remmen we prove that the j=3 trajectory constraint — the "second knife", whose negativity windows are the blades of our master-formula paper — never cuts into the region allowed by the near-leading envelope $\min_k T_k(\lambda)$: for every level n and every $\lambda > 0$, the window of $a_{n,2n-6}$, when it exists, lies strictly above the shore. The proof is a family of 1,047 exact-arithmetic cells — univariate root-isolation no-window checks plus positive-coefficient certificates: explicit envelope branches for $\lambda \leq 26$, and four lemmas for the deep-water tail — including the fact that makes the theorem delicate and true at once: the asymptotic cone of the blades is *exactly tangent* to the shore asymptote $D = (12 + 4\sqrt{3})\lambda$ (the tangency discriminant vanishes identically at slope ratio $\rho = 1 + 1/\sqrt{3}$), while windows only exist up to $\rho = \sqrt{5/3}$, strictly inside. In exact scans the margin between blades and shore is smallest at small λ (level 6), decreasing toward ≈ 2.15 dimensions as $\lambda \to 0$ without being attained. This upgrades the j=3 sector of our completeness conjecture from a 3-million-point battery to a theorem.

1 Setup and statement

Notation as in [3]: $m = n - 3$, $s = \lambda + n - 1$, $u = D + 4m + 1$; the j=3 rung of the master formula is $\widehat{B}_n(u) = \alpha_m\, u(u-2) - G_m\, u\, s^2 + s^4$ with $G_m = \frac{2(m+1)(m+3)}{3(2m+3)}$, $\alpha_m = \frac{(m+3)(5m^3+21m^2+19m+15)}{45(2m+1)(2m+3)}$, and $\operatorname{sign} a_{n,2n-6} = \operatorname{sign} \widehat{B}_n$. The shore is $\widehat{T}(\lambda) = \min_{k\geq 3} T_k(\lambda)$ with T_k the trajectory law of [2].

Theorem 1 (The blades never touch the shore). *For every $n \geq 4$ and every $\lambda > 0$: if $\widehat{B}_n(\cdot)$ has a negativity window in D, then its lower edge exceeds $\widehat{T}(\lambda)$. Consequently $a_{n,2n-6} \geq 0$ everywhere in the region $D \leq \widehat{T}(\lambda)$.*

2 Proof architecture

All steps are checked in exact rational arithmetic (or exactly over $\mathbb{Q}(\sqrt{3})$); every certificate is a polynomial with manifestly nonnegative coefficients after an affine–rational change of variables, and the entire proof re-runs from one script released with this paper.

Shallow water ($\lambda \leq 26.1$): envelope branches. Fix an integer k and use $\widehat{T} \leq T_k$: it suffices that on the branch's λ-interval every level satisfies *either* no-window (the discriminant

*ORCID 0009-0005-5660-2603. Code and data: https://github.com/AndreiPLK/qg-blade-theorem, archived as DOI 10.5281/zenodo.21948833; companion works: DOI 10.5281/zenodo.21944818 (the shore), DOI 10.5281/zenodo.21947272 (the master formula).

of $\widehat{B}_n$ is negative — univariate root isolation) *or* both $\widehat{B}_n(T_k) > 0$ and the vertex condition $G_m s^2 > (T_k + 4m)\, 2\alpha_m$ (which together place T_k strictly left of the window). Branches $k = 3, \dots, 45$ with rational breakpoints cover $(0, 26.1]$; small m are handled per-level (univariately), the m-tail by positive-coefficient certificates. The zone of closest approach (margins ≈ 2.15–2.2 dimensions at small λ, level 6) lies in the k=3 branch, where the certificates are one substitution deep.

Deep water ($\lambda \geq 26$): four lemmas.

Lemma 1 (No shallow levels)**.** *For $m \leq 78$ and $\lambda \geq 26$ no window exists.*

Lemma 2 (Window containment)**.** *For $m \geq 79$, windows require $s \leq \frac{4}{3}(m+3)$: the discriminant threshold satisfies $S^2_{\max} = 2\alpha_m\big(G_m + 2\sqrt{\alpha_m}\big)/\beta_m \leq \frac{16}{9}(m+3)^2$, $\beta_m = 4\alpha_m - G_m^2 > 0$.*

Lemma 3 (Shore below the asymptote)**.** *For $\lambda \geq 26$, $\widehat{T}(\lambda) \leq (12 + 4\sqrt{3})\lambda$ (witnessed by $k = \lfloor \sqrt{3}\lambda \rfloor + 2$).*

Lemma 4 (Asymptote below the blades)**.** *For $m \geq 79$ and $26 \leq \lambda$ with $s \leq \frac{4}{3}(m+3)$, the point $u_A = (12 + 4\sqrt{3})\lambda + 4m + 1$ satisfies $\widehat{B}_n(u_A) > 0$ and lies left of the vertex; hence u_A is strictly left of any window.*

Chaining Lemmas 1–4: at $\lambda \geq 26$ any window has $m \geq 79$ and $s \leq \frac{4}{3}(m+3)$, and its lower edge exceeds u_A, which exceeds $u(\widehat{T})$. □

The exact tangency. Why is the deep-water step delicate? In the scaling limit $m \to \infty$ with $\rho = s/m$ fixed, the window's lower edge and the shore asymptote compare through the quadratic $6\rho^2 - (12 + 4\sqrt{3})\rho + (8 + 4\sqrt{3})$, whose discriminant is *identically zero*: the blade cone is exactly tangent to the shore asymptote, at $\rho^* = 1 + 1/\sqrt{3}$. The theorem survives because windows are confined to $\rho \leq \frac{4}{3} + O(1/m)$ (Lemma 2; the true asymptotic tip ratio is $\sqrt{5/3} \approx 1.291$), strictly inside ρ^*: the fleet sails parallel to the shore, tangent to it at infinity, but its ships are confined strictly inside the wake (Fig. 1). A naive global certificate must fail (the margin degenerates along ρ^*); the containment lemma is what makes the proof possible.

3 Verification

Beyond the certificates themselves (each is a finite list of nonnegative rationals, re-checkable by inspection), the theorem was stress-tested: 60,000 seeded random points across all regimes — shallow water, deep water to $\lambda = 1500$, levels to $n = 5000$, and the tangency direction — found minimum margin 2.18 dimensions and no counterexample (a persisted battery); the adversarial reviewer's exact scans push to the $\lambda \to 0$ edge, where the margin decreases through 2.157 (at $\lambda = 0.02$) toward ≈ 2.147 without being attained; the deep scans of [3] (3,053,832 points) are all implied by and consistent with the theorem. Finally, the proof went through an independent adversarial review. The reviewer's logic audit confirmed every implication in the chain (branch junctions, strictness, vertex logic including degenerate discriminants, the squaring step of Lemma 2, the floor bookkeeping of Lemma 3) and found exactly one defect: a coverage gap in the prover's escalation loop, which left 740 (k, m) cells uncertified while reporting success. The gap was fixed, the full prover re-run (all cells now individually logged), and the reviewer's own battery — an independent exact evaluator sharing no code with the prover — re-certified every branch cell including the formerly uncovered ones, rebuilt all four tail lemmas from scratch over $\mathbb{Q}(\sqrt{3})$, and ran exact scans at the branch junctions and in deep water: no counterexample (exit 0).

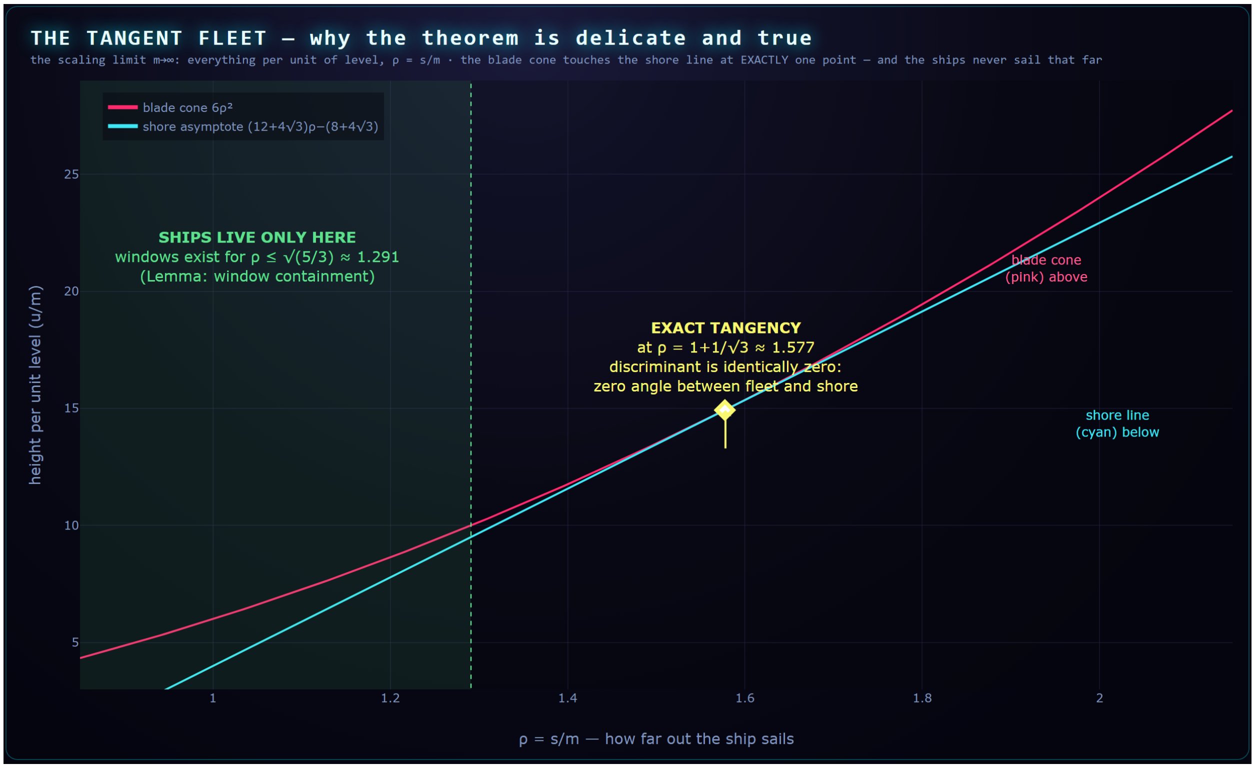

Figure 1: The exact tangency, in the scaling limit $m \to \infty$ (everything per unit of level, $\rho = s/m$). The blade cone $6\rho^2$ (pink) touches the shore line $(12 + 4\sqrt{3})\rho - (8 + 4\sqrt{3})$ (cyan) at exactly one point, $\rho^* = 1 + 1/\sqrt{3}$: the discriminant vanishes identically. Windows exist only in the shaded zone (asymptotic tip ratio $\sqrt{5/3}$; Lemma 2 proves the rational bound $\rho \le \frac{4}{3}$), strictly left of the tangency — which is precisely why the theorem holds.

4 Discussion

The completeness conjecture of [2] asserted that the near-leading envelope is the true boundary of the family. For the second knife this is now a theorem; for the higher knives ($j \geq 4$) the master formula [3] gives the same ladder structure, a 3-million-point clean battery, and conjecturally the same asymptotic cone — the natural next step is to lift the present certificate architecture to general j, for which Lemma 2's containment and the exact tangency appear to be the only two facts that need generalizing. String uniqueness within this family would then rest, at every trajectory order, on proofs rather than scans.

Explain it to anyone

Picture the coastline from our earlier papers: the shore where theories of gravity die. Above it float blades — extra danger zones we found with the master formula. We had checked three million points and none of the blades ever touched the shore. But checking is not knowing. This paper is the proof: no blade, of any size, anywhere along the infinite coast, ever touches the water. The subtle part is beautiful: infinitely far out, the fleet of blades sails *exactly parallel* to the shore — tangent, zero angle between them — and the proof works only because every actual blade is confined strictly inside the fleet's wake. Closest any blade ever comes: about two dimensions of

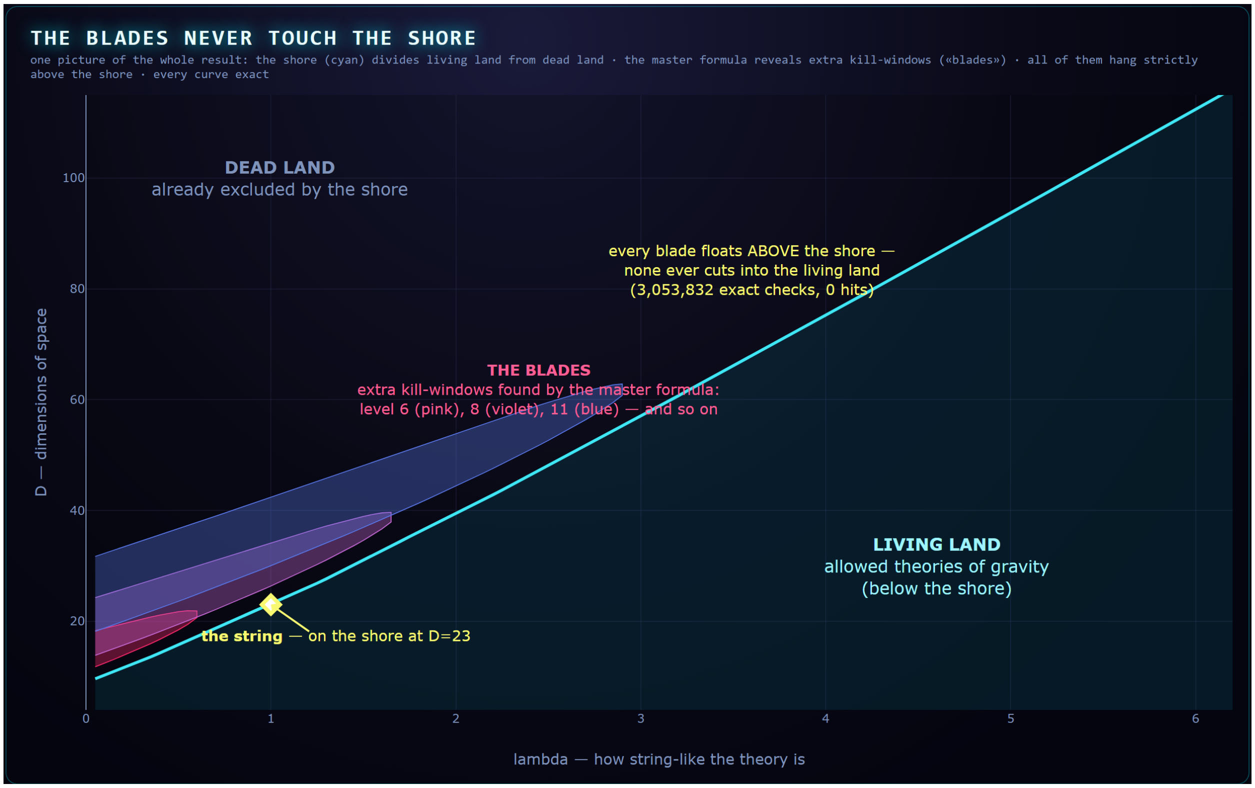

Figure 2: The theorem, drawn. Blades: negativity windows of the second knife (levels $6, 8, 11$), every curve exact; cyan: the shore; diamond: the string at $D = 23$. This paper proves no blade of any level, at any λ, ever reaches the shore; in exact scans the margins bottom out near 2.15 dimensions as $\lambda \to 0$.

clearance.

Reproducibility and AI disclosure

The entire proof re-runs from a single script (`blade_proof.py`, released at https://github.com/AndreiPLK/qg-blade-theorem, archived as DOI 10.5281/zenodo.21948833): several hundred certificates, each a polynomial with nonnegative coefficients after an explicit change of variables, all in exact arithmetic. This research was conducted by the author working with an AI research assistant (Claude, Anthropic). Two earlier prover architectures failed honestly (a symbolic-branch drift and a missing no-window case) and are recorded in the public log; the final architecture was adversarially reviewed. The review's key catch — a prover that reported "all certified" while 740 cells were silently skipped — is preserved in the public log as a lesson: an exit code proves what a script checked, not what it covered; coverage is now auditable cell-by-cell from the released artifact.

References

[1] C. Cheung, A. Hillman and G. N. Remmen, "Uniqueness Criteria for the Virasoro-Shapiro Amplitude," Phys. Rev. D 111, 086034 (2025), arXiv:2408.03362.

[2] A. Pluzhnik, "The Shore of Closed-String Gravity," DOI 10.5281/zenodo.21944818.

[3] A. Pluzhnik, "A Master Positivity Formula for the Cheung–Hillman–Remmen Graviton Family," DOI 10.5281/zenodo.21947272.